Math 275D Lecture 14 Notes

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1 Donsker's Theorem

1.1 Donsker's theorem

Let S_m^n be a simple random walk ($1 \leq m \leq n$). $f^{(n)}(t)$ is defined with S_m^n by linearly interpolating between the values and rescaling. Then $f^{(n)} : [0,1] \to \mathbb{R}$ is a random function.

Theorem 1.1 (Donsker). $f^{(n)} \xrightarrow{d} B(\cdot)$.

Remark 1.1. That is, we can consider $\mathbb{P}_{f^{(n)}}$, which is a measure on $C([0,1])$. This converges weakly to $\mathbb{P}_{B(\cdot)}$ as a measure on $C([0,1])$. When we talk about weak convergence of measures, we mean $\mu_n(A) \to \mu(A)$ for all open, measurable A. Say Brownain motion is associated to the space $(\Omega, \mathcal{F}, \mathbb{P})$. Then the limiting measure here is $\mathbb{P}|_{\mathcal{F}(C([0,1]))}$.

This only talks about convergence of the distributions as measures on $C([0,1])$. We can see that $|\{x_0 : \frac{d}{dx} f^{(n)}(x_0) \text{ exists}\}| = 1$, but $|\{x : B'(x) \text{ exists}\}| = 0$. The issue is that $\{\omega : [g'(t_0)](\omega) \text{ exists}\}$ is not an open set in $C([0,1])$.

1.2 Applications of Donsker's theorem

Corollary 1.1. *If $\psi : C([0,1]) \to \mathbb{R}$ is continuous $\mathbb{P}_0$-a.s. (which is the $\mathbb{P}$ for Brownian motion), then $\psi(f_n) \xrightarrow{d} \psi(B)$.*

Corollary 1.2. *Let $M(f) = \max_t f(t)$. Then $M(f_n) \xrightarrow{d} M(B)$.*

Proof. M is continuous on $C([0,1])$. $\square$

Corollary 1.3. *Let $P_0(f) = |\{x : f(x) > 0\}|$. Then $P_0(f_n) \xrightarrow{d} P_0(B)$.*

Remark 1.2. P_0 is not continuous in $C([0,1])$.

Proof. The set of points where P_0 is continuous has probability 1 for $\mathbb{P}_{\text{BM}}$. Indeed, if $|\{x : g(x) = 0\}| = 0$, then P_0 is continuous at g. $\square$

Similarly, we can use Donsker's theorem to find the last zero of Brownian motion in $[0,1]$.

1.3 Proof of Donsker's theorem

Here is the proof of Donsker's theorem. The idea is to "grow a simple random walk on Brownian motion."

Proof. Let $S_m^n = \sum_{k=1}^m X_k$. We know that $f_n(t) \overset{d}{\approx} B(t)$ for $t = k/n$, but this is hard to deal with. The correct idea is to look at $n^{-1/2} S_m^n = n^{-1/2} \sum_{k=1}^m X_k = B(\tau_1^n + \cdots + \tau_m^n)$, where $\tau_{i+1} = \inf\{t - \tau_i > 0 : |\tilde{B}(t) - \tilde{B}(\tau_i)| \geq 1\}$. Depending on whether the Brownian motion goes up or down, we can tell the random walk to go up or down.

Here is the idea of how the convergence works: Now all $f^{(n)}$ grow on the same B. So the $f^{(n)}$ share good properties of B. If $\mathcal{A}_{\varepsilon,\delta} = \{B : |x - y| \leq \delta \implies |B(x) - B(y)| \leq \varepsilon\}$. In $\mathcal{A}_{\varepsilon,\delta}$, for all large enough n, f_n is ε, δ-continuous. $\square$